On exponential stabilizability with arbitrary decay rate for linear systems in Hilbert spaces

Xavier Dusser, Rabah Rabah

▶ To cite this version:

Xavier Dusser, Rabah Rabah. On exponential stabilizability with arbitrary decay rate for linear systems in Hilbert spaces. Systems Analysis Modelling Simulation, 2000, 37 (4), pp.417-433. hal-00830745

HAL Id: hal-00830745

https://hal.science/hal-00830745

Submitted on 5 Jun 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On exponential stabilizability with arbitrary decay rate for linear systems in Hilbert spaces

Xavier Dusser & Rabah Rabah*

Institut de Recherche en Cybernétique de Nantes
1, rue de la Noë
BP 92101
F-44321 Nantes Cedex 3, France
E-mail: dusser@emn.fr, rabah@emn.fr

Abstract: In this paper, we deal with linear infinite dimensional systems in Hilbert spaces. In the beginning, systems are continuous. In the second part, we use a sampling to transform our first system in a discrete-time system. For these systems, exact controllability implies that the extended controllability gramian is uniformly positive definite. This operator allows us to define a feedback control law which stabilizes exponentially the closed-loop system. In the first case, we want to push eigenvalues in the negative part of the complex plane and to have an arbitrary decay rate. In the discrete-time system, we want to push eigenvalues of the closed loop system in the unit ball that is to say to minimize the norm of the closed loop operator. Results of continuous-time system are applied to a system described by a wave equation.

Keywords: Infinite dimensional systems, Linear systems, Continuous-time systems, Discrete-time systems, Exact controllability, Stabilizability.

*Corresponding author. Also with École des Mines de Nantes, 4, rue A. Kastler, BP 20722, F-44307, Nantes Cedex 3

1 Introduction

Consider the continuous-time system described by the equation:

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & Ax(t)+Bu(t)\\ y(t) & = & Cx(t)\end{array}\right. \tag{1}$$

with initial condition:

$$x(0)=x_0 \tag{2}$$

where $x(t)$ and $u(t)$ take value respectively in Hilbert spaces X and U. A is a linear bounded operator and is the infinitesimal generator of a $\mathcal{C}_0$-group $S(t)$, $t\in\mathbb{R}$, of linear continuous operator on X. B is a linear bounded operator.
This system is a general representation of several physical systems described by partial differential equation or differential equation with delays. Several papers and books were devoted to the study of control problems for this type of systems (see [3] and references given there). The problem of exact controllability is now well known and studied. This property implies exponential stabilizability with arbitrary decay rate. This result was in fact given in [7], where a feedback control law was given. Komornik in [4] uses a feedback control law which is look like ours, with an unbounded control operator B. It is well known that the property of exact controllability for infinite dimensional systems depends on the time of control T. The feedback law given by Slemrod or by Komornik depends also on this time which may be calculated but it is, in general, a complicated problem. Our purpose is to avoid this difficulty and to give a simpler way to design the feedback law.

Often, if digital computation is involved, we need to work with discrete-time versions which arise if the inputs and outputs are sampled at one fixed rate. Consider the discrete-time system described by the equation

$$\left\{\begin{array}{rcl}x_{k+1} & = & A_dx_k+B_du_k\\ y_k & = & Cx_k\end{array}\right. \tag{3}$$

with initial condition x_0. This discrete-time system can be deduce from the continuous-time system (1) and we will see relations in theorem 3.1. Several papers and books are devoted to the study of this type of system (see for example [1] or [2]). Properties of exact controllability and complete stabilizability are well known and studied. From now, a lot of results are writing with a fixed state feedback. In this paper, we describe a state feedback law which can stabilize completely the system (3) and fix arbitrarily the dimension of the ball containing eigenvalues.

The first section is devoted to continuous-time system. The subsection 2.1 of this paper is devoted to exact controllability of the system (1). After the

definition, we consider a necessary and sufficient condition for exact controllability of the system which was also given in [6] as another different criterion of controllability using the operator $N(\lambda)$.
After, we determine a linear operator F from the inverse of the operator $N(\lambda)$ such that the control system with feedback control law $u(t) = Fx(t)$ stabilizes the system. Unlike Slemrod in [7], we can impose the decay rate of the system. Finally, we apply result for a simple system modeled by the wave equation. We show that the system is exactly controllable and we calculate a feedback law. Afterwards, we verify the stability of the closed-loop system. We also verify that our operator allow simpler results than those with Slemrod's operator.

The second section is devoted to discrete-time system. After some definition about exact controllability, we define an extended gramian of controllability K_λ. The end of this section is about complete stabilizability. We define an operator F generated from the inverse of the operator K_λ. We verify the stability of the closed loop system with the feedback law $u_k = Fx_k$.

2 Continuous-time systems

The mild solution of system (1), (2) is given by:

$$x(t, x_0, u) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau.$$

Let $\omega_0(-A)$ be the scalar defined by:

$$\omega_0(-A) = \lim_{t\to+\infty} \left(\frac{1}{t}\ln\|S(t)\|\right).$$

This constant is called the growth bound of the semi-group. Therefore, we can defined the decay rate of the semi-group. For all $\omega > \omega_0(-A)$, there exists a positive constant M_ω such that for all $t \geq 0$, $\|S(t)\| \leq M_\omega e^{\omega t}$. If $\omega_0(-A) = -\infty$ then $\omega \in \mathbb{R}$ may be chosen arbitrarily.

Let $N(\lambda)$ be the bounded operator defined by:

$$N(\lambda) = \int_0^\infty e^{-\lambda t} S(-t)BB^*S^*(-t)xdt.$$

This operator is defined for all positive scalar λ such that $\lambda > 2\omega_0(-A)$ (cf [6]) and is called the extended controllability gramian. This operator is a generalization of the operator $N_T(\lambda)$ defined in [7] by:

$$N_T(\lambda) = \int_0^T e^{-\lambda t} S(-t)BB^*S^*(-t)xdt,$$

with $T < \infty$. This operator was used to design a stabilizing feedback in [7].

In the sequel, $\langle .,. \rangle$ and $\|.\|$ denote respectively the inner product and the norm in the adequate space.

2.1 Exact controllability

Definition 2.1 *[5] A system is said to be exactly controllable if there exists a positive time T such that for all $x_0, x_1 \in X$ and for some control u, we have:*

$$x(T) = x(T, x_0, u) = x_1.$$

A necessary and sufficient condition for exact controllability is given by:

$$\int_0^T \|B^* S^*(\tau)x\| d\tau \geq \delta_T \|x\|^2 \tag{4}$$

for some $\delta_T > 0$ and for all $x \in X$. The condition (4) is verified if and only if the operator K_T defined by:

$$K_T x = \int_0^T S(\tau)BB^*S^*(\tau)x d\tau.$$

is uniformly positive definite that is to say that there exists some $\delta_T > 0$ such that:

$$\langle K_T x, x\rangle \ \geq \delta_T \|x\|^2, \forall x \in X.$$

The operator K_T is the controllability gramian.

The precedent criterion of controllability depends on the time T which is not defined *a priori*. In the next proposition, we have a necessary and sufficient condition where the time T does not appear.

Proposition 2.2 *[6] The system (1), (2) is exactly controllable if and only if the operator $N(\lambda)$ is a uniformly positive definite operator, that is:*

$$\langle N(\lambda)x, x\rangle \ \geq \delta \|x\|^2, \forall x \in X, \delta > 0. \tag{5}$$

Let us note that this result may be formulated for a semi-group which is not a group (see [5]).
If the relation (5) holds true then the operator $N(\lambda)$ admits a bounded inverse $N(\lambda)^{-1}$.

2.2 Complete stabilizability

For the beginning, we precise the definition of complete stabilizability.

Definition 2.3 *[5] The system (1), (2) is said to be completely stabilizable if for all $\omega \in \mathbb{R}$, there exist a linear bounded operator $F : X \to U$ and a constant $M > 0$ such that the semi-group generated by $A + BF$, say $S_F(t)$, verifies:*

$$\|S_F(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

Exact controllability implies complete stabilizability. The converse does not always hold true. In particular conditions, complete stabilizability implies exact controllability. In our case, as A generates a group, it is true because of the next theorem.

Theorem 2.4 *[8] If system (1), (2) is completely stabilizable and the operator generates a group of operators $S(t), t \in \mathbb{R}$, then system (1), (2) is exactly controllable in some time $T > 0$.*

We now recall a property of the operator $N(\lambda)$.

Proposition 2.5 *[6] The operator $N(\lambda)$ maps the domain of definition $D(A^*)$ of the operator A^* into the domain of definition $D(A)$ of the operator A. Moreover, if $x \in D(A^*)$, then the following relation holds:*

$$AN(\lambda)x + N(\lambda)A^*x + \lambda N(\lambda)x = BB^*x \tag{6}$$

After this recall of the operator $N(\lambda)$, we can use it to define a feedback control law which can stabilize our system.

Theorem 2.6 *Let the system (1), (2) be exactly controllable. Let the operator F be defined by:*

$$F = -B^*N^{-1}(\lambda).$$

Then the closed-loop system with $u = Fx$ is exponentially stable. Moreover by mean of the choice of λ, the decay rate may be arbitrarily chosen:

$$\forall \omega \in \mathbb{R}, \exists \lambda, \|S_F(t)\| \leq M_\omega e^{\omega t}. \tag{7}$$

Proof: The closed loop system with $u = Fx$ can be written:

$$\begin{cases} \dot{x}(t) &= \left(A - BB^*N^{-1}(\lambda)\right) x(t) \\ x(0) &= x_0 \end{cases}$$

With equation (6), we have:

$$\begin{array}{ll} AN(\lambda) + N(\lambda)A^* + \lambda N(\lambda) = BB^* & \text{in } D(A^*) \\ AN(\lambda) - BB^* = -N(\lambda)A^* - \lambda N(\lambda) & \text{in } D(A^*) \\ A - BB^*N^{-1}(\lambda) = -N(\lambda)A^*N^{-1}(\lambda) - N(\lambda)\lambda N^{-1}(\lambda) & \text{in } D(A) \\ A - BB^*N^{-1}(\lambda) = N(\lambda)[-A - \lambda I]^*N^{-1}(\lambda) & \text{in } D(A) \end{array}$$

Let $\bar{A} = A - BB^*N^{-1}(\lambda)$ and $\tilde{A} = [-A - \lambda I]^*$. Then there exists an operator $P = N(\lambda)$ such that:

$$\bar{A} = P\tilde{A}P^{-1}.$$

Operators $\bar{A}$ and $\tilde{A}$ could be said to be "similar". Let $\bar{A}_\mu = \mu\bar{A}(\mu I - \bar{A})^{-1}$ and $\tilde{A}_\mu = \mu\tilde{A}(\mu I - \tilde{A})^{-1}$. Then, we have:

$$\lim_{\mu\to\infty} \|\bar{A}_\mu x - \bar{A}x\| = 0, \forall x \in D(\bar{A})$$

and

$$\lim_{\mu\to\infty} \|\tilde{A}_\mu x - \tilde{A}x\| = 0, \forall x \in D(\tilde{A}).$$

We can deduce that the operators $\bar{A}_\mu$ and $\tilde{A}_\mu$ are "similar" and that the same holds true for their generated semi-groups.
If we take the limit when $\mu \to \infty$, then:

$$\begin{array}{rcl} \lim\limits_{\mu\to\infty} e^{\bar{A}_\mu t} & = & \lim\limits_{\mu\to\infty} [N(\lambda)e^{\tilde{A}_\mu t}N^{-1}(\lambda)] \\ e^{\bar{A}t} & = & N(\lambda)e^{\tilde{A}t}N^{-1}(\lambda) \end{array}.$$

Now, if we take the norm, we obtain:

$$\begin{array}{rcl} \|e^{\bar{A}t}\| & = & \|N(\lambda)e^{\tilde{A}t}N^{-1}(\lambda)\| \\ & \leq & \|N(\lambda)\|.\|e^{\tilde{A}t}\|.\|N^{-1}(\lambda)\| \\ & \leq & \|N(\lambda)\|.\|e^{(-A^*-\lambda I)t}\|.\|N^{-1}(\lambda)\| \\ & \leq & \|N(\lambda)\|.e^{-\lambda t}\|e^{-A^*t}\|.\|N^{-1}(\lambda)\| \end{array}$$

But from [3], there exist two scalars M and α such that:

$$\|e^{-A^*t}\| \leq Me^{\alpha t}.$$

The operator $N(\lambda)$ is bounded and positive definite. Therefore, there exist two positive scalars c_1 and c_2 such that:

$$\|N(\lambda)\| \leq c_1, \quad \|N(\lambda)^{-1}\| \leq c_2.$$

Then, we have:

$$\begin{array}{rcl} \|e^{\bar{A}t}\| & \leq & c_1c_2Me^{\alpha t}e^{-\lambda t} \\ & \leq & c_1c_2Me^{(\alpha-\lambda)t}, \quad \forall\lambda. \end{array}$$

Now, we can see that the semi-group which is generated by $\bar{A} = A - BB^*N^{-1}(\lambda)$ is exponentially stable for all $\lambda > \alpha$. Therefore, the closed-loop system with the feedback control $u = Fx$ is exponentially stabilizable with $F = -B^*N^{-1}(\lambda)$. We can write the relation (7) with $\omega = \alpha - \lambda$ and $M_\omega = c_1c_2M$. ∎

The theorem 2.6 is the converse of the theorem 2.4 edited by Zabczyk. In fact, he says that a completely stabilizable system whose operator A generates a group is exactly controllable. For us, with the assumption that A generates a group, an exactly controllable system is completely stabilizable with the feedback law $u = -B^*N^{-1}(\lambda)$.

2.3 Example

Let us consider the system described by the wave equation:

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(t,r) = a\dfrac{\partial^2 z}{\partial r^2}(t,r) + u(t,r) \\ z(t,0) = z(t,1) = 0 \end{cases}$$

In first, we transform the system to obtain a new one in the form (1). After, we verify that the semi-group is a group which is an important condition. Therefore, we can show the controllability, calculate the feedback and verify the stability of the closed loop system. At last, we compare results with operators $N(\lambda)$ and $N_T(\lambda)$.

Let $x = \begin{pmatrix} z \\ \tilde{z} \end{pmatrix}$, then the wave equation may be rewritten as

$$\dot{x} = \begin{pmatrix} 0 & I \\ a\dfrac{\partial^2}{\partial r^2} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u$$

Therefore, we have:

$$A = \begin{pmatrix} 0 & I \\ a\dfrac{\partial^2}{\partial r^2} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

Let $\left\{ \phi_k(r) = \begin{pmatrix} \phi_{1,k}(r) \\ \phi_{2,k}(r) \end{pmatrix}, k \in \mathbb{Z}^* \right\}$ be the eigenvectors of the operator A and let $\{\lambda_k, k \in \mathbb{Z}^*\}$ be the corresponding eigenvalues.

For all $k \in \mathbb{Z}^*$, we have $A\phi_k(r) = \lambda_k \phi_k(r)$, which gives

$$\begin{cases} \phi_{2,k}(r) = \lambda_k \phi_{1,k}(r) \\ a\dfrac{\partial^2 \phi_{1,k}(r)}{\partial r^2} = \lambda_k \phi_{2,k}(r) \end{cases}$$

and then

$$\begin{cases} \phi_{2,k}(r) = \lambda_k \phi_{1,k}(r) \\ a\dfrac{\partial^2 \phi_{1,k}(r)}{\partial r^2} = \lambda_k^2 \phi_{1,k}(r) \end{cases}$$

We search eigenvectors $\{\psi_n(r), n \in \mathbb{N}^*\}$ of $a\dfrac{\partial^2}{\partial r^2}$ and eigenvalues $\{\sigma_n, n \in \mathbb{N}^*\}$ associated.

Because of initial conditions of the systems, eigenvectors $\{\psi_n(r), n \in \mathbb{N}^*\}$ could be written (see [3]):

$$\psi_n(r) = \alpha \sin(\omega r) + \beta \cos(\omega r)$$

As $\psi_n(0) = \psi_n(1) = 0$, then $\omega = n\pi$ and $\beta = 0$ and

$$\begin{aligned} \psi_n(r) &= \alpha \sin(n\pi r) \\ a\frac{\partial^2 \psi_n(r)}{\partial r^2} &= -an^2\pi^2 \psi_n(r) \end{aligned}$$

Therefore we have:

$$\sigma_n = -an^2\pi^2, n \in \mathbb{N}^*$$
$$\psi_n(r) = \alpha \sin(n\pi r), n \in \mathbb{N}^*$$

Hence, we obtain for the operator A:

$$\lambda_k = jk\pi\sqrt{a}, k \in \mathbb{Z}^*$$
$$\phi_{1,k}(r) = \alpha \sin(k\pi r), k \in \mathbb{Z}^*$$
$$\phi_{2,k}(r) = \lambda_k \alpha \sin(k\pi r), k \in \mathbb{Z}^*$$

that we can also write:

$$\lambda_k = jk\pi\sqrt{a}, k \in \mathbb{Z}^*$$
$$\phi_k(r) = \alpha \begin{pmatrix} \sin(k\pi r) \\ \lambda_k \sin(k\pi r) \end{pmatrix}, k \in \mathbb{Z}^*$$

The eigenvalues of the operator A are simple and the closure of $\{\lambda_k, k \in \mathbb{Z}^*\}$ is totally disconnected. The eigenvectors $\{\phi_k, k \in \mathbb{Z}^*\}$ form a Riesz basis. Hence, A is a Riesz operator.

Let $\{\psi_k, k \in \mathbb{Z}^*\}$ be the eigenvectors of A^* such that $\langle \phi_k, \psi_l \rangle = \delta_{kl}$. Hence, $\{\psi_k, k \in \mathbb{Z}^*; \phi_l, l \in \mathbb{Z}^*\}$ form a biorthogonal basis.
The $\mathcal{C}_0$-group $S_A(t)$ generated by A can be written:

$$\begin{aligned} S_A(t) &= \sum_{k \in \mathbb{Z}^*} e^{\lambda_k t} \langle ., \psi_k \rangle \phi_k \\ &= \sum_{k \in \mathbb{Z}^*} e^{jk\pi\sqrt{a}t} \langle ., \psi_k \rangle \phi_k \end{aligned}$$

As the operator B is defined by:

$$B = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

therefore the operator B^* is defined by:

$$B^* = \begin{pmatrix} 0 & I \end{pmatrix}.$$

The operator $N(\lambda)$ is defined for all λ such that:

$$\lambda > 2\omega_0(-A) = 2 \lim_{t \to \infty} \frac{\ln \|S(-t)\|}{t}.$$

In the other hand

$$\begin{aligned} \|S(-t)\|^2 &= \frac{1}{\|x\|^2} \langle S(-t)x, S(-t)x \rangle \\ &= \frac{1}{\|x\|^2} \langle \sum_{k \in \mathbb{Z}^*} e^{-\lambda_k t} \langle \sum_{l \in \mathbb{Z}^*} e^{-\lambda_l^* t} \langle x, \psi_l \rangle \phi_l, \psi_k \rangle \phi_k, x \rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\|x\|^2}\langle \sum_{k\in\mathbb{Z}^*} e^{-(\lambda_k+\lambda_k^*)t}\langle x,\psi_k\rangle\phi_k, x\rangle \\
&= \frac{1}{\|x\|^2}\langle \sum_{k\in\mathbb{Z}^*} \langle x,\psi_k\rangle\phi_k, x\rangle \\
&= \frac{\|x\|^2}{\|x\|^2} \\
&= 1
\end{aligned}
$$

Therefore we have: $2\omega_0(-A) = 0$. Then, $N(\lambda)$ is definite for all strictly positive λ. This operator may be precisely computed:

$$
\begin{aligned}
N(\lambda)x &= \int_0^{+\infty} e^{-\lambda t} S_A(-t)BB^*S_A^*(-t)x dt \\
&= \int_0^{+\infty} e^{-\lambda t}\sum_{k\in\mathbb{Z}^*} e^{-\lambda_k t}\langle BB^*\sum_{l\in\mathbb{Z}^*} e^{-\lambda_l^* t}\langle x,\psi_l\rangle\phi_l,\psi_k\rangle\phi_k dt \\
&= \int_0^{+\infty} e^{-\lambda t}\sum_{k\in\mathbb{Z}^*} e^{-\lambda_k t}\sum_{l\in\mathbb{Z}^*} e^{-\lambda_l^* t}\langle x,\psi_l\rangle\langle BB^*\phi_l,\psi_k\rangle\phi_k dt \\
&= \int_0^{+\infty} e^{-\lambda t}\sum_{k\in\mathbb{Z}^*} e^{-(\lambda_k+\lambda_k^*)t}\langle x,\psi_k\rangle\phi_k dt \\
&= \int_0^{+\infty} e^{-\lambda t}\sum_{k\in\mathbb{Z}^*} \langle x,\psi_k\rangle\phi_k dt \\
&= \left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^{+\infty}\sum_{k\in\mathbb{Z}^*} \langle x,\psi_k\rangle\phi_k \\
&= \frac{1}{\lambda}\sum_{k\in\mathbb{Z}^*} \langle x,\psi_k\rangle\phi_k \\
N(\lambda)x &= \frac{x}{\lambda}
\end{aligned}
$$

Then, we have:

$$
\langle N(\lambda)x, x\rangle = \frac{1}{\lambda}\|x\|^2.
$$

Therefore, there exists a positive constant δ such that: $\langle N(\lambda)x, x\rangle = \delta\|x\|^2$. The system is exactly controllable from proposition 2.2.

As $N(\lambda) = \frac{1}{\lambda}$, then we have $N(\lambda)^{-1} = \lambda$. Therefore, the closed-loop state F can be written: $F = -\lambda B^*$. The closed-loop system is given by $\dot{x} = \hat{A}x$, with

$$
\hat{A} = \begin{pmatrix} 0 & I \\ a\frac{\partial^2}{\partial r^2} & -\lambda \end{pmatrix}.
$$

From now on, we can calculate the eigenvalues $\{\hat{\lambda}_k, k\in\mathbb{Z}^*\}$ of the closed-loop system. The eigenvectors $\{\phi_k, k\in\mathbb{Z}^*\}$ are the same as those of operator A. For all $k\in\mathbb{Z}^*$, the relation $\hat{A}\phi_k(r) = \hat{\lambda}_k\phi_k(r)$ gives:

$$
\begin{aligned}
\phi_{2,k}(r) &= \hat{\lambda}_k\phi_{1,k}(r) \\
a\frac{\partial^2\phi_{1,k}(r)}{\partial r^2} - \lambda\phi_{2,k}(r) &= \hat{\lambda}_k\phi_{2,k}(r)
\end{aligned}
$$

which implies

$$\phi_{2,k}(r) = \hat{\lambda}_k \phi_{1,k}(r)$$
$$a\frac{\partial^2 \phi_{1,k}(r)}{\partial r^2} - \lambda\hat{\lambda}_k\phi_{1,k}(r) - \hat{\lambda}_k^2\phi_{1,k}(r) = 0$$

Because of boundary conditions, the eigenvectors $\{\phi_{1,k}(r), k \in \mathbb{Z}^*\}$ can be written:

$$\phi_{1,k}(r) = \alpha \sin(k\pi r)$$

We obtain for all $k \in \mathbb{Z}^*$:

$$-ak^2\pi^2\phi_{1,k}(r) - \lambda\hat{\lambda}_k\phi_{1,k}(r) - \hat{\lambda}_k^2\phi_{1,k}(r) = 0$$
$$\hat{\lambda}_k^2 + \lambda\hat{\lambda}_k + ak^2\pi^2 = 0.$$

Finally,

$$\hat{\lambda}_k = -\lambda \pm ik\pi\sqrt{a}, \quad k \in \mathbb{Z}^*$$

and

$$\phi_k(r) = \alpha \begin{pmatrix} \sin(k\pi r) \\ \hat{\lambda}_k \sin(k\pi r) \end{pmatrix}, \quad k \in \mathbb{Z}^*.$$

As $k \neq 0$, the eigenvalues of the closed-loop system have their real part strictly negative. The closed-loop system is stable.
If we take the operator $N_T(\lambda)$ instead of $N(\lambda)$, we obtain:

$$N_T(\lambda) = \frac{1 - e^{-\lambda T}}{\lambda}$$

and then we would have a more complicated feedback. The eigenvalues of the closed-loop system are also complex. Moreover, we have use the time T which is unknown. The operator $N(\lambda)$ is really simpler than the other.

3 Discret-time systems

Relations between the system (1) and the system (3) are defined in the next theorem:

Theorem 3.1 *[1] The exact discrete linear system of the system (1) sampled with a BOZ of frequency $1/T$ is defined by the representation (3) with*

$$\begin{aligned} x_k &= x(kT) \\ A_d &= e^{AT} \\ B_d &= \left(\int_0^T e^{At}\right) B \end{aligned}$$

The solution of system (3) is given by:

$$x_k = A_d^k x_0 + \sum_{i=0}^{k-1} A_d^{k-1-i} B_d u_i \tag{8}$$

where x_0 designed the initial state.

Let A_λ the operator defined by:

$$A_\lambda = A_d e^{\lambda T}, \quad \lambda > 0.$$

We chose λ so that $\|A_\lambda^{-1}\| < 1$ that is to say

$$\begin{array}{c} \|A_d^{-1} e^{-\lambda T}\| < 1 \\ \|A_d^{-1}\| e^{-\lambda T} < 1 \\ \|A_d^{-1}\| < e^{\lambda T} \\ \lambda > \frac{ln\|A_d^{-1}\|}{T} \end{array}$$

3.1 Exact controllability

Definition 3.2 *A system is said to be exactly controllable if for all $x \in X$, for all initial condition $x_0 \in X$ there exists a scalar k such that for some control $u_i, i = 1, \ldots, k$, x_k verify equation (8) and $x_k = x$.*

From the next definition, we can deduce that the system is exactly controllable if and only if

$$\bigcup_{i=0}^{\infty} Im(\begin{array}{cccc} B_d & A_d B_d & \ldots & A_d^k B_d \end{array}) = X. \tag{9}$$

We can show, with the open mapping theorem, that the relation (9) is equivalent to

$$\exists k \text{ such that } Im(\begin{array}{cccc} B_d & A_d B_d & \ldots & A_d^k B_d \end{array}) = X.$$

Equivalently, the gramian of controllability K defined by:

$$K = \sum_{i=0}^{k} A_d^i B_d B_d^* A_d^{*i}$$

must be positive definite.

As continuous-time systems, the definition of an extended gramian of controllability will allow us to defined a new criterion of controllability. Let K_λ this gramian defined by:

$$K_\lambda = \sum_{i=0}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$$

Theorem 3.3 *Suppose A_d nonsingular.*
The system (3) is exactly controllable if and only if the operator K_λ is positive definite.

Proof:
We must show that the two gramian K and K_λ are equivalent. As A_d is invertible, K is positive definite if and only if

$$\sum_{i=0}^{k} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$$

positive definite. Now, we must show that $\sum_{i=0}^{k} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$ positive definite iff $\sum_{i=0}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$ is positive definite. One implication is easy, we only show the converse.
Suppose that the operator $\sum_{i=0}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$ is positive definite. There exists a $\delta > 0$ such that

$$\langle \sum_{i=0}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle > \delta \|x\|^2.$$

Then

$$\begin{aligned}\langle \sum_{i=0}^{k-1} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle &= \langle \sum_{i=0}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle - \langle \sum_{i=k}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle \\ &\geq \delta \|x\|^2 - \langle \sum_{i=k}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle\end{aligned}$$

We know that

$$\langle \sum_{i=k}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle \leq |\sum_{i=k}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}|.\|x\|^2$$

As the series $\sum_{i=k}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$ converges, for all $\epsilon > 0$, there exists a number M such that for all $k > M$, we have:

$$|\sum_{i=k}^{\infty} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}| < \epsilon$$

So, we obtain that

$$\langle \sum_{i=0}^{k} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x\rangle \geq (\delta - \epsilon)\|x\|^2$$

As we can choose a K such that $\delta > \epsilon > 0$, we have

$$\langle \sum_{i=0}^{k} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i} x, x \rangle \geq (\delta - \epsilon)\|x\|^2 > 0$$

The operator $\sum_{i=0}^{k} A_\lambda^{-i} B_d B_d^* A_\lambda^{*-i}$ is positive definite. ∎

3.2 Complete stabilizability

Proposition 3.4 *The extended gramian of controllability K_λ is the solution of the Lyapunov equation*

$$A_\lambda^{-1} K_\lambda A_\lambda^{*-1} = K_\lambda - B_d B_d^* \tag{10}$$

PROOF:

$$\begin{aligned} K_\lambda &= B_d B_d^* + A_\lambda^{-1} B_d B_d^* A_\lambda^{*-1} + A_\lambda^{-2} B_d B_d^* A_\lambda^{*-2} + \dots \\ &= B_d B_d^* + A_\lambda^{-1}[B_d B_d^* + A_\lambda^{-1} B_d B_d^* A_\lambda^{*-1} + \dots] A_\lambda^{*-1} \\ &= B_d B_d^* + A_\lambda^{-1} K_\lambda A_\lambda^{*-1} \end{aligned}$$

∎

Theorem 3.5 *[2] The system (3) is stable if and only if the eigenvalues of A are strictly less than one in magnitude (and strictly inside the unit circle in the complex plane).*

Theorem 3.6 *The system (3) is supposed to be controllable.*
Let F the operator defined by:

$$F = -B_d^* K_\lambda^{-1} A_\lambda.$$

The closed loop system can be written

$$x_{n+1} = \tilde{A} x_n$$

with

$$\tilde{A} = A_\lambda - B_d B_d^* K_\lambda^{-1} A_\lambda$$

The operator $\tilde{A}$ is stable. Moreover, for all $\delta < 1$, it is possible to find a positive λ such that

$$\|\tilde{A}\| < \delta < 1.$$

PROOF:
Using the equation (10), we have:

$$\begin{array}{l} A_\lambda^{-1}K_\lambda A_\lambda^{*-1} = K_\lambda - B_dB_d^* \\ A_\lambda^{-1}K_\lambda A_\lambda^{*-1}K_\lambda^{-1} = I - B_dB_d^*K_\lambda^{-1} \\ A_\lambda^{-1}K_\lambda A_\lambda^{*-1}K_\lambda^{-1}A_\lambda = A_\lambda - B_dB_d^*K_\lambda^{-1}A_\lambda \\ (A_\lambda^{-1}K_\lambda)A_\lambda^{*-1}(A_\lambda^{-1}K_\lambda)^{-1} = A_\lambda - B_dB_d^*K_\lambda^{-1}A_\lambda \end{array}$$

Let $A_1 = A_\lambda^{*-1}$ and $A_2 = A_\lambda - B_dB_d^*K_\lambda^{-1}A_\lambda$. Then there exists an operator $P = A_\lambda^{-1}K_\lambda$ such that:

$$A_1 = PA_2P^{-1}.$$

Operators A_1 and A_2 could be said "similar".
As we defined:

$$A_\lambda = A_de^{\lambda T}, \quad \lambda > \frac{ln\|A_d^{-1}\|}{T},$$

we have

$$\|A_\lambda^{*-1}\| \leq \|A_d^{*-1}\|e^{-\lambda T} < 1.$$

As the operator P is positive definite and bounded, there exist two positive scalars c_1 and c_2 such that: $\|P\| \leq c_1, \qquad \|P^{-1}\| \leq c_1.$
Then,we have also

$$\|A_\lambda - B_dB_d^*K_\lambda^{-1}A_\lambda\| \leq c_1c_2\|A_d^{*-1}\|e^{-\lambda T}$$

If we want to have

$$\|\tilde{A}\| \leq \delta < 1,$$

we must have

$$c_1c_2\|A_d^{*-1}\|e^{-\lambda T} \leq \delta.$$

Finally, we have:

$$\lambda > \frac{1}{T}ln(\frac{c_1c_2\|A_d^{*-1}\|}{\delta})$$

∎

4 Conclusion

The result given in this paper is quite general in the case of a bounded control operator B. Moreover, we have seen on the example that the result is easily applicable. This method may be also applied for the case of unbounded operator B.

For discrete-time systems, results are similar. The state feedback law is easily applicable as in the case of the continuous-time systems.

References

[1] d'Andréa-Novel B., Cohen de Lara M. (1994), *Commandes linéaires des systèmes dynamiques*, MASC, Masson.

[2] Balakrishan A.V. (1983), *Elements of state space theory of systems*, Optimization Software, INC, New-York.

[3] Curtain R.F., Zwart H.J. (1995), *An introduction to infinite dimensional linear systems theory*, Springer-Verlag.

[4] Komornik V. (1995), *Stabilisation rapide de problèmes d'évolution linéaires*, C. R. Acad. Sci. Paris, t. 321, Série 1, pp 581-586.

[5] Rabah R., Karrakchou J. (1997), *Exact controllability and complete stabilizability for linear systems in Hilbert spaces*, Appl. Math. Lett., Vol. 10, No. 1, pp 35-40.

[6] Sklyar G.M. (1991), *On exact controllability for differential equation with unbounded operator*, (in Russian), Vestnik Kharkov. Universiteta, PMM 361, pp 20-27.

[7] Slemrod M. (1974), *A note on complete controllability and stabilizability for linear control systems in Hilbert space*, SIAM J. Control, Vol. 12, No. 3, pp 500-508.

[8] Zabczyk J. (1992), *Mathematical control theory: an introduction*, Birkhäuser, Boston.